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Plethysms of Schur functions and the shell model

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Abstract

We present a method for evaluating plethysms of Schur functions that is conceptually simpler than existing methods. Moreover the algorithm can be easily implemented with an algebraic computer language. Plethysms of sums, differences and products of S-functions are dealt with in exactly the same manner as plethysms of simple S-functions. Sums and differences of S-functions are of importance for the description of multi-shell configurations in the shell model. The number of variables in which the S-functions are expressed can be specified in advance, significantly simplifying the calculations in typical applications to many-body problems. The method relies on an algorithm that we have developed for the product of monomial symmetric functions. We present a new way of calculating the Kostka numbers (using Gel'fand patterns) and give, as well, a new formula for the Littlewood–Richardson coefficients.

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1. Introduction

In group theoretical models of many-particle systems, of particular importance is the construction of N-body states that simultaneously belong to definite irreducible representations (irreps) of both the model's relevant group G and the symmetric group S_N. For instance, wavefunctions that describe N fermions must be totally antisymmetric with respect to any permutation of the N fermions; hence to construct states of well defined permutation symmetry one considers S_N as a subgroup of the chain of groups relevant to the model. Mathematically this requires the decomposition of tensor power representations of G into representations that have a particular symmetry type with respect to particle permutations; this operation on the group characters is known as the symmetrized power or *plethysm*.

The groups of interest in many-body physics are the classical compact Lie groups (general linear, special linear, unitary, special unitary, orthogonal, special orthogonal and unitary

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symplectic), as well as the non-compact symplectic Lie groups, and of course the symmetric groups.

Schur functions (S-functions for short) are characters of the unitary irreps of unitary groups (and the characters of finite-dimensional irreps of general linear groups) [1]. The plethysm of S-functions was introduced by Littlewood [2], who developed a number of useful techniques for its calculation [1]. Plethysm of S-functions applies directly to problems involving irreps of unitary groups. However, it is well known [3] that the characters of the unitary irreps of all the other aforementioned Lie groups are expressible in terms of S-functions, and vice versa. To calculate a plethysm for any such group G, then, one expands the characters of G in terms of S-functions, determines the plethysm of the S-functions, and then expresses the resulting series of S-functions back in terms of the characters of G. This procedure is straightforward in principle for compact groups, although the calculations may be difficult in practice. For non-compact groups, where the characters are infinite series of S-functions, the difficulties are even greater. Methods for evaluating plethysms of, for example, the fundamental irreps of non-compact symplectic groups have been given in the literature [4–8]. These make use of the generating functions of S-function series and enable one, in principle, to obtain the full expansion of such plethysms, though in practice one has to truncate them at a prescribed cutoff. In any case, the plethysm of S-functions is of fundamental importance for the calculation of plethysms of irreps for any of the aforementioned groups.

The practical difficulties in calculating plethysms have stimulated a continuing search for algorithms that are both simple and efficient. Some breakthroughs include a notable paper by Butler and King [9], in which they obtain recurrence relations for plethysms; the algorithm for plethysm of Chen *et al* [10]; and the recently published method of Yang [11] for evaluating the coefficient of a single S-function in the expansion of a plethysm of two S-functions. Also computer codes exist to evaluate S-function plethysms; for example 'SCHUR'[4].

If the plethysm of simple S-functions is arduous in practice additional difficulties arise when one needs to calculate symmetrized tensor powers of a sum, difference or product of S-functions, which we call here for brevity *compound S-functions*. Since plethysm is not distributive on the left, the evaluation of such plethysms has up to now relied upon complicated manipulations of S-functions as expressed in equations (44)–(46) of section 5. Physical situations where plethysms of compound S-functions are required include:

(i) Identification of nuclear shell model states that span irreps of appropriate unitary groups and have a particular permutation symmetry for a system of particles in a multi-shell configuration.
(ii) Classification of many-particle states, of well defined permutation symmetry, by representation characters that are not simple S-functions; for example, by characters of irreps of the orthogonal or unitary symplectic groups.

The objective of this paper is to outline a method for evaluating plethysms of S-functions that is conceptually simpler than existing methods. The method is simpler because it only requires a straightforward algorithm for multiplying monomial symmetric functions and no other complicated rules intervene. Moreover the algorithm can be easily implemented with an algebraic computer language such as Maple. A major strong point of the method is that plethysms of compound S-functions are dealt with in exactly the same way as plethysms of simple S-functions. The number of variables in which the S-functions are expressed can be specified in advance, significantly simplifying the calculations in typical applications to

[4] SCHUR$^{\mathrm{TM}}$ An interactive program for calculating properties of Lie groups and symmetric functions distributed by Steven M Christensen and Associates, Inc., PO Box 16175, Chapel Hill, NC 27516, USA. E-mail: steve@smc.vnet.net, webpage http://smc.vnet/Christensen.html

many-body problems. In essence, the method described here consists of (i) converting the S-functions to monomial symmetric functions, (ii) calculating the plethysm of monomial symmetric function series, where distributivity on the left is valid, and (iii) converting the result back to S-functions. The feasibility of this method relies on an algorithm (presented in section 4) for the product of monomial symmetric functions, which to the best of our knowledge has not previously appeared in the literature.

The structure of the paper is as follows. In section 2 we present the essential facts about partitions and symmetric functions and establish the notation used. In section 3 we review the interrelations among monomial symmetric functions, power-sum symmetric functions and S-functions, and give a novel method for determining the transition matrices for S-functions in terms of monomial symmetric functions using Gel'fand patterns. In section 4 we review the outer product of S-functions and its physical interpretation. We also give an algorithm (developed in detail in appendix B) for the product of two monomial symmetric functions. This algorithm is used in section 4 to resolve the outer product of two S-functions, incidentally obtaining a new formula for the Littlewood–Richardson coefficients, and in section 5 to obtain an algorithm for the resolution of the plethysm of two S-functions. The physical interpretation of plethysms of S-functions is discussed in section 5. Section 6 contains an example illustrating the application of the new algorithm to the plethysm of compound S-functions, and concluding remarks.

2. Partitions and symmetric functions

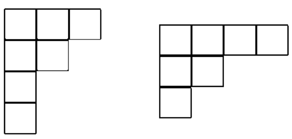
Polynomials in r independent indeterminates $x_1, x_2, \ldots, x_r$ that remain invariant with respect to arbitrary permutations of the indices of the indeterminates are known as *symmetric functions*. Symmetric functions of degree n are labelled by partitions of n.

For further details on symmetric functions and partitions one can consult, for example, the books by Littlewood [1] or Wybourne [3]. A comprehensive source for information on symmetric functions and partitions is Macdonald's book [12]. In this section we shall provide a brief summary for the reader's convenience.

A partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ of the positive integer n is a sequence of positive integers λ_i (the parts of the partition) for which $|\lambda| = \sum_{i=1}^{k} \lambda_i = n$ where $|\lambda|$ denotes the weight of the partition λ. The notation $\lambda \vdash n$ indicates that λ is a partition of n. The partition is said to be standard provided that $\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_k$. The number of non-zero parts of a partition is called the *length* of the partition, $l(\lambda)$.

Exponents, called *multiplicities*, ν_i, are commonly used to simplify the notation for a partition. For example, (4422211111) can be written as $(4^2 2^3 1^5)$, where $\nu_1 = 5$, $\nu_2 = 3$, $\nu_3 = 0$ and $\nu_4 = 2$.

Partitions are easily visualized by using Young diagrams. Specifically, the Young diagram associated with a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ consists of k rows of boxes and has λ_i boxes in its ith row. The main diagonal of a Young diagram consists of the first box in the first row, the second box in the second row and so on. By reflecting the Young diagram for λ in its main diagonal one obtains the Young diagram for its *conjugate* partition λ'. For example, the Young diagrams associated with the partition (321^2) and its conjugate (421) are



(1)

In what follows we will sometimes need to add zeros to the end of a partition in order to have a prescribed number of parts. The resulting partition is equivalent to the original one. For example the partitions (21), (210), (2100) and so on are all equivalent.

In this paper we are only concerned with three types of symmetric functions, the *monomial symmetric functions* m_λ, the *power-sum symmetric functions* p_j and the *S-functions* s_λ. The set of all symmetric functions labelled by $\lambda \vdash n$ forms a vector space for which the sets $\{m_\lambda \mid \lambda \vdash n\}$, $\{p_\lambda \mid \lambda \vdash n\}$ and $\{s_\lambda \mid \lambda \vdash n\}$ are bases.

Consider a fixed number r of variables (indeterminates) $x_1, x_2, \ldots, x_r$; then, the S-function denoted by either[5] s_λ or $\{\lambda\}$ is defined as

$$s_\lambda = \frac{|x_s^{\lambda_t+r-t}|}{|x_s^{r-t}|} \tag{2}$$

where s and t index rows and columns respectively of the $r \times r$ determinants. For example, in terms of three ($r = 3$) indeterminates, the S-function labelled by $\lambda = (21)$ (recall that $(21) \equiv (210)$) is given by

$$s_{(210)} = \frac{\begin{vmatrix} x_1^{2+2} & x_1^{1+1} & x_1^{0+0} \\ x_2^{2+2} & x_2^{1+1} & x_2^{0+0} \\ x_3^{2+2} & x_3^{1+1} & x_3^{0+0} \end{vmatrix}}{\begin{vmatrix} x_1^2 & x_1^1 & x_1^0 \\ x_2^2 & x_2^1 & x_2^0 \\ x_3^2 & x_3^1 & x_3^0 \end{vmatrix}}$$

or

$$s_{(21)} = x_1^2x_2^1x_3^0 + x_1^1x_2^2x_3^0 + x_1^0x_2^2x_3^1 + x_1^0x_2^1x_3^2 + x_1^2x_2^0x_3^1 + x_1^1x_2^0x_3^2 + 2x_1^1x_2^1x_3^1. \tag{3}$$

One of the facts that makes S-functions so useful is that these formulae are essentially independent of the number of variables; the exception is that if there are insufficient variables, then some S-functions are identically zero. To be precise, an S-function with k parts in r variables is identically zero if $r < k$. Equation (2) follows from the Weyl character formula applied to unitary groups, so the characters of unitary irreps of $U(r)$ are in one-to-one correspondence with S-functions in r variables. Also note that an S-function s_λ is said to be *standard* if the partition λ that labels it is a standard partition. However non-standard S-functions can also be defined. A non-standard S-function is either zero (if $\lambda_{i-1} = \lambda_i - 1$) or can be converted to a standard one by using the well known S-function modification rule [3]

$$s_{(\lambda_1,\ldots,\lambda_{i-1},\lambda_i,\ldots,\lambda_k)} = -s_{(\lambda_1,\ldots,\lambda_i-1,\lambda_{i-1}+1,\ldots,\lambda_k)} \tag{4}$$

which is a consequence of the properties of determinants, more precisely the determinant in the numerator of equation (2).

In terms of the r indeterminates, the monomial symmetric function m_λ (m-function for short) is defined as

$$m_\lambda = \sum x_1^{\lambda_1}x_2^{\lambda_2}\cdots x_k^{\lambda_k} \qquad (k \leqslant r) \tag{5}$$

where the label λ stands for the partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, and the sum includes all *distinct* terms obtained by all possible permutations of the subscripts i of the indeterminates x_i. The number of terms in the sum (which henceforth we call the *dimension* $\dim(m_\lambda)$ of the m-function) is given by

$$\dim(m_\lambda) = \frac{r!}{\prod_i \mu_i!} \tag{6}$$

[5] Both notations are widely used in the literature.

where μ_i, for $i \geqslant 1$, are simply the multiplicities of the parts of λ and $\mu_0 = r - k$. Thus, for example, for $r = 3$ and $\lambda \vdash 3$ we have the three m-functions $m_{(3)} = x_1^3 + x_2^3 + x_3^3$, $m_{(21)} = x_1^2x_2 + x_1x_2^2 + x_1^2x_3 + x_1x_3^2 + x_2^2x_3 + x_2x_3^2$ and $m_{(111)} = x_1x_2x_3$, with dimensions $\dim(m_{(300)}) = 3$, $\dim(m_{(210)}) = 6$ and $\dim(m_{(111)}) = 1$.

The power-sum symmetric function p_j is the sum of the jth powers of the r indeterminates:

$$p_j = \sum_{i=1}^{r} x_i^j = m_{(j)}. \tag{7}$$

For example, for $r = 4$ and $j = 3$, $p_3 = x_1^3 + x_2^3 + x_3^3 + x_4^3$. In addition, the p-function p_λ, where $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, is defined as

$$p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_k}. \tag{8}$$

3. Transitions among symmetric functions

In order to implement the plethysm procedure in section 5 we need to calculate elements of the transition matrices that interrelate the above-mentioned bases for symmetric functions. Note that in what follows we assume that the related symmetric functions are expressed in the same variables.

3.1. Expansion of S-functions in terms of p-functions

The remarkable relationship between the unitary and symmetric groups, known as the Schur–Weyl duality, leads to a direct relationship between a character χ^λ of the symmetric group and the corresponding character s_λ for a unitary group. For $\lambda \vdash n$ one has

$$s_\lambda = \sum_\rho \frac{\chi_\rho^\lambda}{z_\rho} p_\rho \tag{9}$$

where $\rho = (\rho_1, \rho_2, \ldots)$ labels the conjugacy classes of S_n, $p_\rho = \prod_i p_{\rho_i}$ are p-functions, χ_ρ^λ are the components of the character χ^λ of the irrep λ of S_n and z_ρ is given [12] by

$$z_\rho = \nu_1!1^{\nu_1}\nu_2!2^{\nu_2}\cdots\nu_n!n^{\nu_n}. \tag{10}$$

3.2. Expansion of S-functions in terms of m-functions

The S-function s_λ is expressed in terms of m-functions as follows:

$$s_\lambda = \sum_{\substack{|\mu|=|\lambda| \\ \mu_1 \leqslant \lambda_1}} K_{\lambda\mu}\, m_\mu \tag{11}$$

where μ is a standard partition of $|\lambda|$ with μ_1 not exceeding λ_1. The coefficients $K_{\lambda\mu}$, which are either positive integers or zero, are known as the Kostka numbers [12]. Several methods are given in the literature for the determination of the Kostka numbers $K_{\lambda\mu}$. Typically they are calculated by combinatorial means involving Young diagrams. The simplest is to list the standard numberings of the relevant Young tableaux with integers in the range $1, \ldots, n$ such that the numbers $(\mu_1, \mu_2, \ldots)$ of occurrences of the integers $(1, 2, \ldots, n)$ are such that $\mu_{i+1} \leqslant \mu_i$; e.g., for the $U(3)$ S-function $s_{(21)}$ the list contains the numbered tableaux

$$\begin{array}{|c|c|}\hline 1 & 1 \\ \hline 2 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 \\ \cline{1-1}\end{array} \qquad \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 \\ \cline{1-1}\end{array} \tag{12}$$

corresponding to $\mu = (210)$, (111) and (111), respectively. This gives

$$s_{(21)} = m_{(21)} + 2m_{(111)}. \tag{13}$$

In appendix A we explain the physical meaning of Kostka numbers in terms of representation theory. However, for computational purposes, it is more efficient to identify and enumerate the partitions μ in (11) by means of Gel'fand patterns.

For a given S-function (in r indeterminates), labelled by $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$, the m-functions m_μ appearing on the rhs of (11) can be identified by means of triangular arrays of integers (the Gel'fand patterns) [13].

$$\begin{array}{ccccccccccc}
g_{11} & & g_{12} & & g_{13} & & \cdots & & g_{1r-1} & & g_{1r} \\
& g_{21} & & g_{22} & & g_{23} & & \cdots & & g_{2r-1} & \\
& & g_{31} & & g_{32} & & \cdots & & g_{3r-2} & & \\
& & & & & \vdots & & & & & \\
& & & & & g_{rr} & & & & &
\end{array} \tag{14}$$

whose entries are subject to the following conditions:

$$\begin{aligned}
g_{1i} &= \begin{cases} \lambda_i & \text{for} \quad i \leqslant k \\ 0 & \text{for} \quad k < i \leqslant r \end{cases} \\
g_{k-1,i} &\geqslant g_{ki} \geqslant g_{k-1,i+1}.
\end{aligned} \tag{15}$$

The parts of each partition μ are given by differences of the sums of the entries in two successive rows of the array

$$\begin{aligned}
\mu_i &= \sum_{l=1}^{r-i+1} g_{il} - \sum_{l=1}^{r-i} g_{i+1,l} \\
\mu_r &= g_{rr}.
\end{aligned} \tag{16}$$

It is clear that for a partition λ there are many compatible patterns. In fact, there are as many as $\sum_\mu \dim(m_\mu)$. In other words, each possible Gel'fand pattern gives one term of each m-function that comprises s_λ; equivalently, the Gel'fand patterns are in one-to-one correspondence with the semistandard Young tableaux of shape λ. However one does not need to construct all of the Gel'fand patterns in order to identify the relevant m-functions. It is enough to recognize the distinct patterns that give rise to leading terms, i.e. those for which $\mu_i \geqslant \mu_{i+1}$. This selection can be efficiently implemented by imposing on the entries of these patterns the extra condition

$$\sum_{l=1}^{r-i+1} g_{il} - \sum_{l=1}^{r-i} g_{i+1,l} \geqslant \sum_{l=1}^{r-i} g_{i+1,l} - \sum_{l=1}^{r-i-1} g_{i+2,l}. \tag{17}$$

Consider, for example, the S-function $s_{(21)}$ in three indeterminates. The triangular patterns that satisfy the required conditions are

$$\begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 0 & \\ & & 0 & & \end{array} \qquad \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 2 & & 0 & \\ & & 1 & & \end{array} \qquad \begin{array}{ccccc} 2 & & 1 & & 0 \\ & 1 & & 1 & \\ & & 1 & & \end{array} \quad . \tag{18}$$

Thus, $s_{(21)} = m_{(21)} + 2m_{(111)}$.

3.3. Expansion of m-functions in terms of S-functions

An m-function m_λ is expressed in terms of S-functions as

$$m_\lambda = \sum_{\sigma \vdash |\lambda|} K^{-1}_{\lambda\sigma} s_\sigma \tag{19}$$

where the coefficients $K^{-1}_{\lambda\sigma}$, which can be either positive or negative integers, are the inverse Kostka numbers. The number of terms in the sum (19) cannot exceed the dimension of m_λ and depends on the number of indeterminates.

It may appear that the coefficients $K^{-1}_{\lambda\sigma}$ can be obtained trivially by inverting the appropriate Kostka matrix. However that method is rather uneconomical in the context of plethysm calculations because one typically requires a particular row of the inverse of a Kostka matrix that does not necessarily need to be constructed (since the S-functions in the final result have higher degree than the original S-function factors). A practical, easily programmable procedure for determining the inverse Kostka numbers requires only two straightforward steps conveyed by the following formula[6]:

$$m_\lambda = \sum_{\hat{\lambda}} s_{\hat{\lambda}} = \sum_\sigma K^{-1}_{\lambda\sigma} s_\sigma \tag{20}$$

where the first sum runs over all distinct partitions $\hat{\lambda}$ generated from λ by permuting its parts in all possible ways. Clearly, only one of the partitions $\hat{\lambda}$ is standard. Application of the modification rules to the non-standard S-functions leads to s_σ and the sought-after inverse Kostka numbers.

As an example, consider the m-function $m_{(321)}$ in three indeterminates. We have that

$$m_{(321)} = s_{(321)} + s_{(312)} + s_{(231)} + s_{(213)} + s_{(123)} + s_{(132)}. \tag{21}$$

Since, by the modification rules,

$$\begin{aligned} s_{(312)} &= s_{(231)} = s_{(123)} = 0 \\ s_{(213)} &= -s_{(222)} \\ s_{(132)} &= -s_{(222)} \end{aligned} \tag{22}$$

then the resulting S-function expansion (in three indeterminates) is

$$m_{(321)} = s_{(321)} - 2s_{(222)}. \tag{23}$$

4. Products of symmetric functions

4.1. Physical interpretation of S-function products

Each operation involving S-functions corresponds to an operation on unitary representations of $GL(n)$ or its subgroups [16]. In particular, outer products of S-functions correspond to tensor products of unitary irreps of $U(n)$ or $GL(n)$[7]. To see that outer products of S-functions have fundamental physical importance for descriptions of many-particle states consider the following. Suppose that a state of N_p protons is specified by s_λ and a state of N_n neutrons is specified by s_μ, so that $\lambda \vdash N_\mathrm{p}$ and $\mu \vdash N_\mathrm{n}$. Then a state of the combined system of $N_\mathrm{p} + N_\mathrm{n}$ particles is specified by an S-function s_σ that occurs in the expansion of the outer product

$$s_\lambda s_\mu = \sum_\sigma \Gamma_{\lambda\mu\sigma} s_\sigma \tag{24}$$

where each partition σ is a partition of $N_\mathrm{p} + N_\mathrm{n}$. Note that all of the S-functions s_λ, s_μ and s_σ are irreps of the same unitary group $U(n)$.

Another S-function operation that is of importance in physical applications is plethysm or symmetrized power. Consider single-particle states labelled by s_λ corresponding to an irrep of

[6] Note that this procedure arises from multiplying an m-function by a Vandermonde determinant and rearranging the result as discussed in [1, 14]. We would like to thank a referee for pointing out that this procedure can be traced back at least as far as Muir in 1882; see [15, pp 150–1].

[7] The results obtained are independent of n, and are therefore particularly powerful.

$U(n)$; then states of N identical particles with permutation symmetry $[\nu]$, $\nu \vdash N$, are labelled by an S-function s_σ occurring in the expansion of the outer plethysm[8]

$$s_\lambda \otimes s_\nu = \sum_\sigma \Lambda_{\lambda\nu\sigma} s_\sigma. \tag{25}$$

In equation (25), ν labels an irrep of S_N and s_σ are irreps of $U(n)$. More details about this operation and the new algorithm to evaluate it are deferred until section 5.

4.2. Product of m-functions

The product of m-functions is a simple product of polynomials. In order to develop an efficient algorithm for this product without having to work out all the terms explicitly, let us first define the *addition of two partitions* $(\alpha_1, \alpha_2, \ldots)$ and $(\beta_1, \beta_2, \ldots)$ as being the partition whose parts are $(\alpha_1+\beta_1, \alpha_2+\beta_2, \ldots)$. It is necessary that the partitions have an equal number of parts; if they do not, then one increases the number of parts of the shortest one by adding enough zeros at the end.

The product of two simple m-functions is then defined as

$$m_\alpha m_\beta = \sum_\gamma I_\gamma m_\gamma \tag{26}$$

where the partitions γ result from adding to α all distinct partitions $\hat{\beta}$ obtained by permuting in all possible ways the parts of β. Note that one reorders the parts of the resulting partitions γ to make them standard. The multiplicity I_γ of a resulting m-function m_γ is given by

$$I_\gamma = n_\gamma \frac{\dim(m_\alpha)}{\dim(m_\gamma)} \tag{27}$$

where n_γ is the number of times the same partition γ appears in the process of adding partitions referred to above. One can determine n_γ as follows. Write α in the form $(k^{\mu_k}(k-1)^{\mu_{k-1}} \cdots 2^{\mu_2} 1^{\mu_1} 0^{\mu_0})$ where the μ_i are the multiplicities defined as in equation (6). Then

$$n_\gamma = P_k P_{k-1} \cdots P_1 P_0 \tag{28}$$

where P_k is the number of distinct permutations of the first μ_k parts of the partition $\hat{\beta}$ (not necessarily in standard form), P_{k-1} is the number of distinct permutations of the next μ_{k-1} parts of $\hat{\beta}$ and so forth, where $\gamma = \alpha + \hat{\beta}$. Clearly, all m-functions on the left or right of equation (26) are functions of the same r indeterminates. If all the possible partitions γ are to appear in the expansion (26) (i.e. the expansion is complete) then r should be set equal to the sum of the lengths of α and β. However, if the product (in equation (26)) is part of a calculation involving irreps of the unitary group $U(n)$ then one should set $r = n$, since S-functions with more than n parts are identically zero in $U(n)$.

As an example consider the product $m_{(32)}m_{(11)}$ and choose $r = 4$ indeterminates so that the resulting expansion is complete. The addition of $\alpha = (3200) \equiv (32)$ to the list (1100), (1010), (1001), (0110), (0101), (0011) (i.e. to the partitions generated from $(1100) \equiv (11)$ by permuting its parts in all possible but distinct ways) gives

$$(4300), (4210), (4201), (3301), (3310), (3211). \tag{29}$$

With the values given in table 1, where use was made of (6), we obtain the final result

$$m_{(3200)}m_{(1100)} = m_{(4300)} + m_{(4210)} + 2m_{(3310)} + m_{(3211)}.$$

[8] Since in this paper we only refer to the outer plethysm we shall drop the adjective 'outer' from here on.

Table 1. Multiplicities of the m-functions in the expansion of the product $m_{(32)}m_{(11)}$ in $r = 4$ indeterminates.

γ	n_γ	$\dim(m_\gamma)$	I_γ
(4300)	1	60	1
(3310)	2	60	2
(4210)	2	120	1
(3211)	1	60	1

Note that if we had chosen $r = 3$ instead (as in the context of $U(3)$), then the calculation would have involved only partitions in no more than three parts; i.e., $\alpha = (320)$ and $\beta = (110)$. In this case, the result obtained is

$$m_{(320)}m_{(110)} = m_{(430)} + m_{(421)} + 2m_{(331)} \tag{30}$$

which, as expected, is the same as (30) except for the absence of the partition with four non-zero parts.

The algorithm for the multiplication of two m-functions (cf equation (26)) can be easily generalized to the multiplication of series of m-functions. The result is (for more details see appendix B)

$$\left(\sum_{i_1} c_{i_1} m_{\alpha_{i_1}}\right)\left(\sum_{i_2} c_{i_2} m_{\alpha_{i_2}}\right)\cdots\left(\sum_{i_p} c_{i_p} m_{\alpha_{i_p}}\right) = \sum_{\gamma_{12,\ldots,p}} \Upsilon_{\gamma_{12,\ldots,p}} m_{\gamma_{12,\ldots,p}} \tag{31}$$

where the coefficients

$$\Upsilon_{\gamma_{12,\ldots,p}} = \sum_{j,i_p} \Upsilon_{\gamma_{12,\ldots,p-1_j,i_p}} \delta_{\gamma_{12,\ldots,p}\{\gamma_{12,\ldots,p-1_j,i_p}\}} \tag{32}$$

can be found recursively. For example

$$\Upsilon_{\gamma_{12}} = \sum_{i_1,i_2} \Upsilon_{\gamma_{i_1,i_2}} \delta_{\gamma_{12}\{\gamma_{i_1,i_2}\}} \tag{33}$$

where $\delta_{\gamma_{12}\{\gamma_{i_1,i_2}\}} = 1$ if the partition γ_{12} appears in the set $\{\gamma_{i_1,i_2}\}$ and zero otherwise. According to equation (27),

$$\Upsilon_{\gamma_{i_1,i_2}} = c_{i_1}\, c_{i_2}\, n_{\gamma_{i_1,i_2}} \frac{\dim(m_{\alpha_{i_1}})}{\dim(m_{\gamma_{i_1,i_2}})} = c_{i_1} c_{i_2} I_{\gamma_{i_1,i_2}} \tag{34}$$

$$\Upsilon_{\gamma_{123}} = \sum_{j,i_3} \Upsilon_{\gamma_{12_j,i_3}} \delta_{\gamma_{123}\{\gamma_{12_j,i_3}\}} \tag{35}$$

with

$$\Upsilon_{\gamma_{12_j,i_3}} = \sum_{j,i_3} \Upsilon_{\gamma_{12}} c_{i_3} n_{\gamma_{12_j,i_3}} \frac{\dim(m_{\gamma_{12_j,i_3}})}{\dim(m_{\gamma_{123}})} \tag{36}$$

and so on.

4.3. *The Littlewood–Richardson coefficients*

The standard method for evaluating the rhs of (24) is the well known Littlewood–Richardson rule [17], which is a set of directives to be applied to the corresponding Young diagrams. Although application of these rules is feasible in simple cases, their execution for partitions of large numbers is rather complex. The resolution, given below, of the outer product of two S-functions via the product of m-functions is simpler and much more amenable to automatic computation. Using the algorithm for the product of m-functions, one has for the outer product of two S-functions

$$s_\mu s_\nu = \left(\sum_{\alpha_i} K_{\mu\alpha_i} m_{\alpha_i}\right)\left(\sum_{\beta_j} K_{\nu\beta_j} m_{\beta_j}\right) \tag{37}$$

$$= \sum_\gamma \Upsilon_\gamma m_\gamma \tag{38}$$

$$= \sum_\sigma \sum_\gamma K^{-1}_{\gamma\sigma} \Upsilon_\gamma s_\sigma \tag{39}$$

where $\Upsilon_\gamma = \Upsilon_{\gamma_{12}}$; cf (33).

Note that we have now obtained a new formula for the Littlewood–Richardson coefficients,

$$\begin{aligned}\Gamma_{\mu\nu\sigma} &= \sum_\gamma K^{-1}_{\gamma\sigma} \Upsilon_\gamma \\ &= \sum_\gamma K^{-1}_{\gamma\sigma} \sum_{ij} \Upsilon_{\gamma_{ij}}\ \delta_{\gamma\{\gamma_{ij}\}} \\ &= \sum_\gamma K^{-1}_{\gamma\sigma} \sum_{ij} K_{\mu\alpha_i} K_{\nu\beta_j} n_{\gamma_{ij}} \frac{\dim(\alpha_i)}{\dim(\gamma_{ij})} \delta_{\gamma\{\gamma_{ij}\}}.\end{aligned} \tag{40}$$

Note also that this formula enables one to calculate the coefficient of a single S-function in the expansion of the outer product without having to construct the whole expansion.

5. Plethysm of S-functions

The plethysm of two S-functions s_λ and s_μ, of weights $|\lambda| = n$ and $|\mu| = k$ respectively,

$$s_\lambda \otimes s_\mu = \sum_{\sigma \vdash nk} \Lambda_{\lambda\mu\sigma} s_\sigma \tag{41}$$

gives a sum of S-functions, all of weight nk, with non-negative integer coefficients $\Lambda_{\lambda\mu\sigma}$. Plethysm is a symmetrized power of S-functions, i.e. the outer product of k copies of s_λ can be decomposed into a sum of sets of terms, where the S-functions in each set have permutation symmetry (μ):

$$\underbrace{s_\lambda s_\lambda \cdots s_\lambda}_{k \text{ copies}} = \sum_\mu f^\mu s_\lambda \otimes s_\mu \tag{42}$$

where the sum on the right-hand side extends over all partitions of k, and f^μ is equal to the dimension of the S_k irrep labelled by μ. In more physical terms, the states of a k-particle system (where each particle is individually described by s_λ) of permutation symmetry $[\mu]$ are described by the plethysm $s_\lambda \otimes s_\mu$.

For example the simple (outer) product $s_1\ s_1\ s_1$ has expansion

$$s_1 s_1 s_1 = s_3 + 2s_{21} + s_{111}. \tag{43}$$

If one regards s_1 as the character of a $U(3)$ irrep spanned by the wavefunctions of a single particle, then the product $s_1\ s_1\ s_1$ is the character of the tensor product of three copies of this irrep. This reducible representation is spanned by a set of three-particle wavefunctions. It is a direct sum of three irreps: an irrep with character s_3 spanned by wavefunctions that are fully symmetric with respect to particle exchange; an irrep with character s_{111} spanned by fully antisymmetric wavefunctions and two mixed symmetry irreps with character s_{21}.

Note that a given S-function may appear in different symmetrized products, so S-functions do not characterize the symmetry classes of tensor products. In contrast with the outer product, plethysm of S-functions is not commutative or distributive on the left over outer product, addition and subtraction.

Current algorithms cannot directly handle plethysms of compound S-functions (i.e. linear combinations or outer products of S-functions). The difficulty is due to the fact that plethysm of S-functions is not distributive on the left and so use must be made of [3]

$$(A + B) \otimes s_\lambda = \sum_{\mu,\nu} \Gamma_{\mu\nu\lambda}(A \otimes s_\mu)(B \otimes s_\nu) \tag{44}$$

$$(A - B) \otimes s_\lambda = \sum_{\mu,\nu} (-1)^{|\nu|}\Gamma_{\mu\nu\lambda}(A \otimes s_\mu)(B \otimes s_{\nu'}) \tag{45}$$

$$(AB) \otimes s_\lambda = \sum_{\mu,\nu} k_{\mu\nu\lambda}(A \otimes s_\mu)(B \otimes s_\nu) \tag{46}$$

where A and B stand for either S-functions, characters of classical groups or any linear combination of S-functions. In equations (44) and (45) the coefficients $\Gamma_{\mu\nu\lambda}$ are taken from $s_\mu s_\nu = \sum_\lambda \Gamma_{\mu\nu\lambda} s_\lambda$ and $s_{\nu'}$ is the S-function labelled by the conjugate partition of ν. In equation (46) the coefficients $k_{\mu\nu\lambda}$ are taken from the internal (inner) product $s_\mu \circ s_\nu = \sum_\lambda k_{\mu\nu\lambda} s_\lambda$, (cf [3]). It is assumed in all three equations that the summations include the cases $s_\mu = s_0 = 1$ and $s_\nu = s_\lambda$, and vice versa.

If use of equations (44)–(46) is already tedious when A and B are simple S-functions, it becomes even more so when they are compound S-functions. In the following we present a method that permits the evaluation of the plethysm of a compound S-function without having to resort to the labour entailed by using equations (44)–(46).

By expanding the S-functions on the left side of the plethysm sign $\otimes$ in terms of m-functions and the S-function on the right side in terms of p-functions, the plethysm of a compound S-function (or of a simple one) is then reduced to the plethysm of a series of m-functions with p-functions. But since the plethysm of an m-function with a p-function is still an m-function,

$$m_\mu \otimes p_j = m_{j.\mu} \tag{47}$$

where $j \cdot \mu$ means that each part of μ is multiplied by j (that is, if $\mu = (\mu_1, \ldots, \mu_k)$ then $j \cdot \mu = (j\mu_1, \ldots, j\mu_k)$), then evaluation of the plethysm of S-functions (simple or compound) only involves, just as for the outer product, the multiplication of m-functions.

5.1. An algorithm for the plethysm of S-functions

Here is a detailed description for evaluating the plethysm

$$s_\lambda \otimes s_\nu = \sum_\sigma \Lambda_{\lambda\nu\sigma} s_\sigma. \tag{48}$$

First, the S-function s_ν is expanded, as usual, in terms of the p-functions $p_\rho = \prod_i p_{\rho_i}$

$$s_\nu = \sum_\rho \chi^\nu_\rho z^{-1}_\rho p_\rho \tag{49}$$

and since plethysm of S-functions is distributive on the right,

$$\begin{aligned} s_\lambda \otimes s_\nu &= \sum_\rho \chi^\nu_\rho z^{-1}_\rho s_\lambda \otimes \prod_i p_{\rho_i} \\ &= \sum_\rho \chi^\nu_\rho z^{-1}_\rho \prod_i s_\lambda \otimes p_{\rho_i}. \end{aligned} \tag{50}$$

Now, the S-function, s_λ, is expanded in terms of m-functions

$$s_\lambda = \sum_j K_{\lambda\alpha_j} m_{\alpha_j} \tag{51}$$

and by making use of the property

$$m_\mu \otimes p_j = m_{j\cdot\mu} \tag{52}$$

one obtains

$$s_\lambda \otimes s_\nu = \sum_\rho \chi_\rho^\nu z_\rho^{-1} \prod_i^{l(\rho)} \left(\sum_j K_{\lambda\alpha_j} m_{\rho_i\cdot\alpha_j} \right). \tag{53}$$

It remains to evaluate the product

$$\prod_i^{l(\rho)} \left(\sum_j K_{\lambda\alpha_j} m_{\rho_i\cdot\alpha_j} \right)$$

which can be done using (32) to yield

$$\prod_{i=1}^{p=l(\rho)} \left(\sum_j K_{\lambda\alpha_j} m_{\rho_i\cdot\alpha_j} \right) = \sum_{\gamma^\rho_{12\ldots p}} \Upsilon_{\gamma^\rho_{12\ldots p}} m_{\gamma^\rho_{12\ldots p}}. \tag{54}$$

Thus,

$$s_\lambda \otimes s_\nu = \sum_\rho \chi_\rho^\nu z_\rho^{-1} \sum_{\gamma^\rho_{12\ldots p}} \Upsilon_{\gamma^\rho_{12\ldots p}} m_{\gamma^\rho_{12\ldots p}} = \sum_\gamma \Upsilon_\gamma m_\gamma \tag{55}$$

with

$$\Upsilon_\gamma = \sum_\rho \chi_\rho^\nu z_\rho^{-1} \Upsilon_{\gamma^\rho_{12\ldots p}} \delta_{\gamma\{\gamma^\rho_{12\ldots p}\}} \tag{56}$$

where the set $\{\gamma\}$ is the union of all sets of partitions $\{\gamma^\rho_{12\ldots p}\}$ for all classes ρ, and the delta function $\delta_{\gamma\{\gamma^\rho_{12\ldots p}\}}$ ensures that the coefficient Υ_γ has contributions from the individual coefficients $\Upsilon_{\gamma^\rho_{12\ldots p}}$ when a partition γ is common to more than one set $\{\gamma^\rho_{12\ldots p}\}$.

Converting the m-functions back to S-functions we finally have

$$s_\lambda \otimes s_\nu = \sum_\sigma \sum_\gamma \Upsilon_\gamma K^{-1}_{\gamma\sigma} s_\sigma. \tag{57}$$

Thus,

$$\Lambda_{\lambda\nu\sigma} = \sum_\gamma \Upsilon_\gamma K^{-1}_{\gamma\sigma}. \tag{58}$$

The plethysm of a sum, difference or outer product of S-functions clearly reduces to the evaluation of an equation formally identical to (53), since a sum of a series of m-functions is still a series of m-functions. For example,

$$(s_\lambda + s_\mu) \otimes s_\nu = \sum_\rho \chi_\rho^\nu z_\rho^{-1} \prod_i^{l(\rho)} \left(\sum_\alpha K_{\lambda\alpha} m_{\rho_i\cdot\alpha} + \sum_\beta K_{\mu\beta} m_{\rho_i\cdot\beta} \right).$$

6. Illustrative example and concluding remarks

To illustrate the method presented in this paper, let us consider the plethysm in $U(3)$

$$(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)} \tag{59}$$

relevant to the physical problem of finding the $U(3)$ representations that can describe three particles placed with permutation symmetry (21) in any of the valence shells $N = 1, 2,$ or 3 of the spherical harmonic oscillator.

We shall outline first how the calculation is carried out using equation (44) (which we refer to as method I), and then discuss the evaluation of the same plethysm by means of the procedure advocated in this paper (method II) and implemented with Maple. Of course the result via method I could also be obtained, in practice, by automatic computation, but the evaluation steps of method II are, in our opinion, simpler and therefore easier to program.

With method I, one applies equation (44) with, for example, $A = s_{(1)} + s_{(2)}$ and $B = s_{(3)}$ to obtain

$$\begin{aligned}(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)} = {} & [(s_{(1)} + s_{(2)}) \otimes s_{(21)}][s_{(3)} \otimes s_{(0)}] + [(s_{(1)} + s_{(2)}) \otimes s_{(0)}][s_{(3)} \otimes s_{(21)}] \\ & +[(s_{(1)} + s_{(2)}) \otimes s_{(2)}][s_{(3)} \otimes s_{(1)}] + [(s_{(1)} + s_{(2)}) \otimes s_{(1)}][s_{(3)} \otimes s_{(2)}] \\ & +[(s_{(1)} + s_{(2)}) \otimes s_{(1^2)}][s_{(3)} \otimes s_{(1)}] + [(s_{(1)} + s_{(2)}) \otimes s_{(1)}][s_{(3)} \otimes s_{(1^2)}]. \end{aligned} \tag{60}$$

Applying equation (44) to the first factor of each term in the preceding equation, one is left with plethysms of 'simple' S-functions which can be evaluated using one's favourite algorithm. The result is sums and products of sums of S-functions. Repeated applications of the Littlewood–Richardson rule then generates the result

$$\begin{aligned}(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)} = {} & s_{(81)} + s_{(72)} + s_{(63)} + s_{(621)} + s_{(54)} + s_{(531)} + s_{(432)} \\ & +s_{(8)} + 2s_{(71)} + 3s_{(62)} + s_{(61^2)} + 3s_{(53)} + 2s_{(521)} + s_{(4^2)} + 2s_{(431)} + s_{(42^2)} + s_{(3^22)} \\ & +2s_{(7)} + 4s_{(61)} + 5s_{(52)} + 2s_{(51^2)} + 4s_{(43)} + 3s_{(421)} + 2s_{(3^21)} + s_{(32^2)} \\ & +2s_{(6)} + 5s_{(51)} + 5s_{(42)} + 2s_{(41^2)} + 2s_{(3^2)} + 3s_{(321)} \\ & +2s_{(5)} + 4s_{(41)} + 3s_{(32)} + 2s_{(31^2)} + s_{(2^21)} + s_{(4)} + 2s_{(31)} + s_{(2^2)} + s_{(21^2)} + s_{(21)}. \end{aligned}$$

Note that the above result is valid for $U(n)$ in general, i.e. for $n \geqslant 3$, since no S-functions in more than three parts appear in the expansion.

With method II we proceed as follows. First $s_{(1)} + s_{(2)} + s_{(3)}$ is expressed in terms of m-functions and $s_{(21)}$ in terms of p-functions,

$$(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)} = (m_{(1)} + m_{(2)} + m_{(1^2)} + m_{(3)} + m_{(21)} + m_{(1^3)}) \otimes \tfrac{1}{3}(p_{(1^3)} - p_{(3)}). \tag{61}$$

Use of equation (47) yields

$$\begin{aligned}(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)} = {} & \tfrac{1}{3}(m_{(1)} + m_{(2)} + m_{(1^2)} + m_{(3)} + m_{(21)} + m_{(1^3)})^3 \\ & -\tfrac{1}{3}(m_{(3)} + m_{(6)} + m_{(3^2)} + m_{(9)} + m_{(63)} + m_{(3^3)}). \end{aligned}$$

The next step is to evaluate the first term on the rhs of the above equation by applying equation (31). At this point the desired number of indeterminates has to be specified. Since we are interested in a $U(3)$ result, we restrict the m-function labels to at most three parts. Finally we collect like terms in the resulting m-function expansion and by means of equation (20) convert them to S-functions[9].

It is clear from this example that the algorithm presented here does not rely on recurrence relations but rather follows very simple rules extremely convenient for automatic computation. The rules of this algorithm are exactly the same whether one has to determine the plethysm of simple or compound S-functions.

The Maple procedure implementing method II of plethysm is

```
> Splethysm:=proc(dim::nonnegint,S1::list(list),S2::list)
local SS2,SS1, SR;
```

[9] Note here that, as mentioned before in this paper, having to construct the Kostka matrix in order to retrieve the required inverse Kostka numbers is not as economical as using equation (20).

```
SS1:=M_content(dim,S1);
SS2:=P_content(S2);
SR:=pleth(dim,SS1,SS2);RETURN(Schur_list(dim,SR));
end:
```

Thus, the command to evaluate the plethysm $(s_{(1)} + s_{(2)} + s_{(3)}) \otimes s_{(21)}$ and the corresponding output are

```
> Splethysm(3,[[1],[2],[3]],[2,1]);

[ [8, 1, 0, 1], [7, 2, 0, 1], [6, 3, 0, 1], [6, 2, 1, 1], [5, 4, 0, 1],
[5, 3, 1, 1], [4, 3, 2, 1],
[8, 0, 0, 1], [7, 1, 0, 2], [6, 2, 0, 3], [6, 1, 1, 1], [5, 3, 0, 3],
[5, 2, 1, 2], [4, 4, 0, 1], [4, 3, 1, 2], [4, 2, 2, 1], [3, 3, 2, 1],
[7, 0, 0, 2], [6, 1, 0, 4], [5, 2, 0, 5], [5, 1, 1, 2], [4, 3, 0, 4],
[4, 2, 1, 3], [3, 3, 1, 2], [3, 2, 2, 1]
[6, 0, 0, 2], [5, 1, 0, 5], [4, 2, 0, 5], [4, 1, 1, 2], [3, 3, 0, 2],
[3, 2, 1, 3]
[5, 0, 0, 2], [4, 1, 0, 4], [3, 2, 0, 3], [3, 1, 1, 2], [2, 2, 1, 1],
[4, 0, 0, 1], [3, 1, 0, 2], [2, 2, 0, 1], [2, 1, 1, 1],
[2, 1, 0, 1] ]
```

Clearly, the entries in the call-command `Splethysm(dim,S1,S2)` are `dim = 3`, where `dim` is the chosen number of indeterminates; `S1 ≡ [[1],[2],[3]]`, a list of lists; each list-element corresponds to an S-function on the lhs of the plethysm operation; `S2 ≡ [2,1]`, a simple list which stands for the S-function on the rhs of the plethysm operation.

Note that the entry-lists do not require the zero parts of the S-functions to be specified but, in the output, the S-functions are labelled by lists with `dim + 1` number of parts. The last element of each list gives the multiplicity of the corresponding S-function.

The procedure `Splethysm` relies on other Maple procedures, namely `M_content`, which expresses the S-functions in terms of m-functions, `P_content`, which expresses an S-function in terms of p-functions, `pleth`, which evaluates the plethysm of m-functions with p-functions, evaluates the product of series of m-functions (whenever necessary) and collects like m-functions together, and finally `Schur_list`, which converts the m-functions back to S-functions.

As with any algorithm for plethysm, run time increases rapidily with the dimensions of the S-functions involved. With a 100 MHz Pentium the plethysm mentioned above takes 12 s. An important point of this method is that one has the ability to establish, *a priori*, the affiliation of the character s_μ (in $s_\mu \otimes s_\nu$), i.e. whether it belongs to $U(2)$ or $U(3)$, etc, so that one can specify the number of indeterminates and considerably simplify the calculations and reduce running time. For example, with the same 100 MHz Pentium, the plethysm $s_{(22)} \otimes s_{(8)}$ takes 11 s for $r = 2$ ($U(2)$) indeterminates, 48.5 s for $r = 3$ ($U(3)$) and about 20 min for $r = 4$ ($U(4)$). Note though that the simplification introduced by establishing from the beginning the maximum number of parts of the resulting S-functions does not result in any loss of accuracy.

In conclusion we have succeeded in giving a method that treats the plethysm of compound S-functions (linear combinations or products of S-functions) on the same footing as the plethysm of simple S-functions. There is no need, in this method, to resort to the use of intricate equations in order to take care of the fact that S-function plethysm is not distributive on the left with respect to addition, subtraction or multiplication. The key point of the method is the fact that the plethysm of a monomial symmetric function by a power-sum symmetric function is still a monomial symmetric function, which does not hold true in general for S-functions. Clearly, the product of two m-functions is a series of m-functions; thus by reducing the S-functions to m-functions the plethysm reduces to a simple multiplication of m-functions, which can be performed without regard for their S-function origin. The S-function content of the plethysm is recovered at the end by converting the final m-functions into S-functions, using again a simple algorithm. The method requires only algorithms for expanding S-functions in terms of m-functions and vice versa, and for evaluating products of series of m-functions. These algorithms are given in sections 3 and 4 respectively. Use of this method is of great advantage for evaluating plethysms of characters of groups other than the unitary groups or plethysms of finite series of S-functions.

Maple procedures to carry out the calculations entailed by these algorithms have been constructed and the whole package will appear shortly in the literature. The procedure to evaluate the outer product of S-functions is also part of the package. Details will be left to the forthcoming publication.

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Appendix A. The Kostka numbers and representation theory

The physical interpretation of Kostka numbers can be made clear in terms of representation theory by means of the following claim.

Claim. If λ is a highest weight for some $U(n)$ irrep then the S-function (character) for this irrep is given by

$$s_\lambda(x) = \sum_{\nu}^{+} K_{\lambda\nu}\, m_\nu(x) \tag{62}$$

where the sum $\sum^+$ is restricted to dominant integral weights and $K_{\lambda\nu}$ is the mulitiplicity of basis states of a weight ν in the irrep.

Note that a dominant integral weight is one having the property that $\nu_1 \geqslant \nu_2 \geqslant \cdots \geqslant \nu_n \geqslant 0$. In the language of Lie algebra structure theory, such a weight belongs to the positive Weyl chamber of weight space.

Proof. An S-function $s_\lambda(x)$ for a unitary group $U(n)$ evaluated at $x = (x_1, \ldots, x_n)$ is the trace over a basis for the irrep of highest weight λ of the matrix $T^\lambda(x)$ representing the diagonal $U(n)$ matrix with diagonal entries $(x_1, \ldots, x_n)$. Under the transformation $T^\lambda(x)$, a state $|\nu\rangle$ of weight $\nu = (\nu_1, \nu_2, \ldots)$ and multiplicity indexed by α transforms as

$$T(x) : |\alpha, \nu\rangle \rightarrow |\alpha, \nu\rangle \times x_1^{\nu_1} x_2^{\nu_2} \ldots . \tag{63}$$

It follows that

$$s_\lambda = \sum_{\alpha,\nu} x_1^{\nu_1} x_2^{\nu_2} \ldots \tag{64}$$

where the sum is over all weights of the irrep. Now observe that, if ν is a weight of a $U(n)$ irrep, the weight obtained by permuting all the parts of $\nu = (\nu_1, \nu_2, \ldots)$, viewed as a partition of $|\lambda|$, is also a weight for the irrep. The set of all weights obtained by permuting the parts of a given weight lie on a Weyl orbit. Such an orbit is characterized by any one weight in the orbit. Moreover, every orbit contains just one weight in the positive Weyl chamber. Next observe that the contribution to the S-function coming from all the weights on a single Weyl orbit is the sum of the distinct terms obtained by permutations $\mathcal{P}$ of the subscripts:

$$m_\nu(x) = \sum_{\mathcal{P}} \mathcal{P}\, x_1^{\nu_1} x_2^{\nu_2} \ldots . \tag{65}$$

It follows that the S-function for an irrep is the sum of such m-functions weighted by the multiplicities $K_{\lambda\nu}$. □

Appendix B. Multiplication of m-functions

The product of two simple m-functions is defined as

$$m_\alpha m_\beta = \sum_\gamma I_{\gamma_{\alpha\beta}} m_\gamma \tag{66}$$

where $\{\gamma\}$ stands for the set of distinct (and ordered) partitions obtained by adding partition α to the partitions derived from β by permuting its parts in all possible, but distinct, ways:

$$I_{\gamma_{\alpha\beta}} = n_\gamma \frac{\dim(m_\alpha)}{\dim(m_\gamma)}. \tag{67}$$

In the case where the product of monomial functions involves multiplicative coeficients, we define

$$c_\alpha m_\alpha \; c_\beta m_\beta = \sum_{\{\gamma\}} \Upsilon_{\gamma_{\alpha\beta}} m_\gamma \tag{68}$$

where

$$\Upsilon_{\gamma_{\alpha\beta}} = c_\alpha c_\beta I_{\gamma_{\alpha\beta}} = c_\alpha c_\beta n_\gamma \frac{\dim(m_\alpha)}{\dim(m_\gamma)}. \tag{69}$$

The product of a sum of two monomial functions by another monomial function is

$$\begin{aligned}(c_{\alpha_1} m_{\alpha_1} + c_{\alpha_2} m_{\alpha_2}) c_\beta m_\beta &= \sum_{\{\gamma_1\}} \Upsilon_{\gamma_{\alpha_1\beta}} m_{\gamma_1} + \sum_{\{\gamma_2\}} \Upsilon_{\gamma_{\alpha_2\beta}}\, m_{\gamma_2} \\ &= \sum_{\{\gamma\}} \Upsilon_{\gamma_{\alpha\beta}} m_\gamma \end{aligned} \tag{70}$$

where the set of partitions $\{\gamma\}$ is the union of the sets $\{\gamma_1\}$ and $\{\gamma_2\}$.

$$\Upsilon_{\gamma_{\alpha\beta}} = \begin{cases} \Upsilon_{\gamma_{\alpha_1\beta}} + \Upsilon_{\gamma_{\alpha_2\beta}} & \text{if} \quad \gamma \in \{\gamma_1\} \bigcap \{\gamma_2\} \\ \Upsilon_{\gamma_{\alpha_1\beta}} & \text{if} \quad \gamma \in \{\gamma_1\} \\ \Upsilon_{\gamma_{\alpha_2\beta}} & \text{if} \quad \gamma \in \{\gamma_2\}. \end{cases} \tag{71}$$

The generalization to a sum of N terms is straightforward:

$$\left(\sum_i c_{\alpha_i} m_{\alpha_i} \right) c_\beta m_\beta = \sum_{\{\gamma\}} \Upsilon_{\gamma_{\alpha\beta}} m_\gamma \tag{72}$$

with

$$\Upsilon_{\gamma_{\alpha\beta}} = \begin{cases} \sum_i \Upsilon_{\gamma_{\alpha_i\beta}} & \text{if} \quad \gamma \in \bigcap \{\gamma_i\} \\ \Upsilon_{\gamma_{\alpha_i\beta}} & \text{if} \quad \gamma \in \{\gamma_i\}. \end{cases} \tag{73}$$

Now the product of two series of monomial functions is

$$\left(\sum_{i_1} c_{\alpha_{i_1}} m_{\alpha_{i_1}}\right)\left(\sum_{i_2} c_{\alpha_{i_2}} m_{\alpha_{i_2}}\right) = \sum_{\{\gamma_{i_1,i_2}\}} \Upsilon_{\gamma_{i_1,i_2}} m_{\gamma_{i_1,i_2}} = \sum_{\{\gamma_{12}\}} \Upsilon_{\gamma_{12}} m_{\gamma_{12}} \tag{74}$$

where

$$\Upsilon_{\gamma_{12}} = \begin{cases} \sum_{i_1,i_2} \Upsilon_{\gamma_{i_1,i_2}} & \text{if} \quad \gamma \in \bigcap\{\gamma_{i_1,i_2}\} \\ \Upsilon_{\gamma_{i_1,i_2}} & \text{if} \quad \gamma \in \{\gamma_{i_1,i_2}\}. \end{cases} \tag{75}$$

Or equivalently,

$$\Upsilon_{\gamma_{12}} = \sum_{i_1,i_2} \Upsilon_{\gamma_{i_1,i_2}} \delta_{\gamma_{12}\{\gamma_{i_1,i_2}\}} \tag{76}$$

where $\delta_{\gamma\{\gamma_{i_1,i_2}\}} = 1$ if the partition γ_{12} appears in the set $\{\gamma_{i_1,i_2}\}$ and zero otherwise.

The product of three series of monomial functions can be obtained by making use of the previous result

$$\left(\sum_{i_1} c_{i_1} m_{\alpha_{i_1}}\right)\left(\sum_{i_2} c_{i_2} m_{\alpha_{i_2}}\right)\left(\sum_{i_3} c_{i_3} m_{\alpha_{i_3}}\right) = \left(\sum_{\{\gamma_{12}\}} \Upsilon_{\gamma_{12}} m_{\gamma_{(12)}}\right) \sum_{i_3} c_{i_3} m_{\alpha_{i_3}}$$

$$\left(\sum_{\{\gamma_{12}\}} \Upsilon_{\gamma_{12}} m_{\gamma_{(12)}}\right) \sum_{i_3} c_{i_3} m_{\alpha_{i_3}} = \sum_{\{\gamma_{12,i_3}\}} \Upsilon_{\gamma_{12,i_3}}\ m_{\gamma_{12,i_3}} = \sum_{\{\gamma_{123}\}} \Upsilon_{\gamma_{123}} m_{\gamma_{123}} \tag{77}$$

with

$$\Upsilon_{\gamma_{123}} = \sum_{i_{12} i_3} \Upsilon_{\gamma_{12,i_3}} \delta_{\gamma_{123}\{\gamma_{12,i_3}\}} \tag{78}$$

$$\Upsilon_{\gamma_{12,i_3}} = \Upsilon_{\gamma_{12}} c_{i_3} n_{\gamma_{12,i_3}} \frac{\dim(m_{\gamma_{12,i_3}})}{\dim(m_{\gamma_{123}})} \tag{79}$$

$$\Upsilon_{\gamma_{i_1,i_2}} = c_{i_1} c_{i_2} n_{\gamma_{i_1,i_2}} \frac{\dim(m_{\alpha_{i_1}})}{\dim(m_{\gamma_{12}})}. \tag{80}$$

Finally the product of p series of monomial functions can be written as

$$\left(\sum_{i_1} c_{i_1} m_{\alpha_{i_1}}\right)\left(\sum_{i_2} c_{i_2} m_{\alpha_{i_2}}\right) \cdots \left(\sum_{i_p} c_{i_p} m_{\alpha_{i_p}}\right) = \sum_{\{\gamma_{12\ldots p}\}} \Upsilon_{\gamma_{12\ldots p}}\ m_{\gamma_{12\ldots p}} \tag{81}$$

where the coefficients

$$\Upsilon_{\gamma_{12\ldots p}} = \sum_{i_{12\ldots p-1,i_p}} \Upsilon_{\gamma_{12\ldots p-1,i_p}} \delta_{\gamma_{12\ldots p}\{\gamma_{12\ldots p-1,i_p}\}} \tag{82}$$

can be found recursively.

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